Intrinsic Bubbles and Fat Tails in Stock Prices

Prasad V. Bidarkota

Department of Economics, University Park DM 320A, Florida International University, Miami, FL 33199, USA; Tel: +1-305-348-6362; Fax: +1-305-348-1524
E-mail address: bidarkot@fiu.edu

Abstract: We study the constant discount rate present value model for stock pricing in a stochastic setting where the exogenous dividend stream is modeled as a random walk with innovations drawn from the family of stable distributions. We derive an exact analytical solution for the fundamental stock price. We evaluate the ability of the model fundamentals and the dividends-driven intrinsic bubbles to explain the observed variation in annual US stock prices. We compare results obtained in this setting with those from the traditional model where all stochastic processes are driven by Gaussian shocks.

Key phrases: Stock prices; present-value model; intrinsic bubbles; fat tails; normal distributions; stable distributions.

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1. INTRODUCTION

Financial models of asset pricing traditionally have not done a very good job of explaining observed variation in stock prices. The failure seems to stem from an inability of pricing models to generate sufficient variation in implied price-dividend ratios. For instance, the constant discount factor present value model with a random walk process for dividends implies a constant price-dividend ratio whereas the observed series shows great fluctuations over time (Leroy and Porter, 1981; Shiller, 1981).

One method to generate non-constant price-dividend ratios in this model is to entertain solutions to asset prices that do not satisfy the transversality condition. Such solutions that are fully rational and depend only on the fundamentals of the model and not on any extraneous factors (such as calendar time) are termed as rational intrinsic bubbles by Froot and Obstfeld (1991).

In the linear present value model with exogenous dividends as the only fundamentals, intrinsic bubbles turn out to be non-linear deterministic functions of dividends. Their non-linear nature allows bubble solutions to generate non-constant price-dividend ratios, and allows them to capture excess sensitivity of stock prices to movements in dividends. However, the non-linear nature of bubbles also implies that they are explosive in nature. With high values of dividends the bubble component of stock prices will be very large. This remains an unsatisfactory feature of bubble solutions to the present value model.
Recent literature suggests that the degree of non-linearity required to generate observed variation in a data series is reduced when one accounts for any fat tails that may exist in the empirical distributions of the data (Bidarkota, 2000). There is a long and celebrated literature documenting fat tails in stock prices, going back to early work by Mandelbrot (1963). McCulloch (1996a) provides a summary of evidence on fat tails in stock prices. In a more recent paper, Lux and Sornette (1999) demonstrate theoretically that stock prices driven by processes with rational bubble components exhibit fat tails.

Mandelbrot (1963) advocated the use of stable distributions for modeling these fat tails. McCulloch (1996a) provides a comprehensive survey on the financial applications of stable distributions. These distributions are the natural extensions of Gaussian distributions, which are widely used on account of their convenience and ease of analytical tractability. Gaussian errors are also often motivated by their Central Limit attributes. If financial markets evolve as an outcome of several individually unimportant decisions of a large number of investors, then one may appeal to the Central Limit Theorem and assume that stock prices can be modeled as Gaussian processes. However, exactly the same argument holds in the stable case as well since the Generalized Central Limit Theorem dictates that the limiting distribution of such a process must belong to the more general class of stable distributions of which the Gaussian is just one member (Zolotarev, 1986).

In this paper, we study rational intrinsic bubbles in the constant discount factor present value model where the only exogenous fundamentals (dividends) evolve as a
random walk stochastic process with innovations that have stable distributions. We derive an exact formula for the present value stock prices in such a setting. We apply this model to analyze annual US stock price data over the last century. We study to what extent the present value stock prices, derived in a stochastic setting that admits fat tails in dividend realizations, matches observed movements in stock prices. We then explore the role of intrinsic bubbles in such a setting. Because our assumed stochastic processes are able to model fat tails in dividends and price data, we expect the contribution of the nonlinear bubble term in rationalizing observed stock prices to be diminished. We compare our results with those obtained within a Gaussian setting that does not account for fat tails.

Driffill and Sola (1998) also study intrinsic bubbles in the present value model, assuming that dividends follow a Markov switching process proposed by Hamilton (1989). They find that the incremental explanatory power of the bubble component over the present value fundamental component is significantly reduced when discrete regime changes are allowed in the mean of the dividends process.

The paper is organized as follows. We set out the present value model for stock prices in section 2 and describe what we mean by the fundamental solution and a bubble solution. In section 3 we derive the fundamental stock price and the bubble under the assumption that dividends evolve as a random walk process with stably distributed innovations. We also compare this solution with that obtained under a Gaussian random walk for dividends. In section 4, we undertake an empirical study to determine to what
extent the present value model, with and without the bubble components, explains the observed variation in stock prices in stable and Gaussian settings. We summarize our main findings in the concluding section.

2. THE PRESENT VALUE MODEL

The present value model with a constant discount rate is given by:

$$P_t = e^{-rt}E_t[D_t + P_{t+1}]. \quad (1)$$

Here, $P_t$ is the real price of a share at the beginning of period $t$

$D_t$ are the real dividends per share paid out over period $t$

$r$ is the non-stochastic and constant discount rate, equal to the real rate of interest

$E_t$ is the mathematical expectation, conditioned on information available at the start of period $t$.

On forward iteration, the present value equation yields:

$$P_t = \sum_{s=t}^{\infty} e^{-r(s-t+1)} E_t(D_s) + \lim_{s \to \infty} e^{-rs} E_t(P_s). \quad (2)$$

One solution to stock prices in the above equation, denoted $P_t^{PV}$, is obtained by imposing the transversality condition:

$$\lim_{s \to \infty} e^{-rs} E_t(P_s) = 0. \quad (3)$$

Imposing the transversality condition on Equation (2) gives:
Thus, this equation provides the fundamental value of the stock price. One specifies an
exogenous stochastic process for dividends and evaluates $P_t^{pv}$.

There exist other solutions to the present value model given in Equation (1) that
do not satisfy the transversality condition in Equation (3). For instance, let $\{B_t\}_{t=0}^{\infty}$ be
any sequence of random variables that satisfy:

$$B_t = e^{-r}E_t \{B_{t+1}\}. \quad (5)$$

One can easily show that $(P_t^{pv} + B_t)$ satisfies Equation (1) but violates Equation (3) for
all $B_t \neq 0$.

If $B_t$ is constructed as a function of the fundamentals alone, i.e. as a function of
the dividends $D_t$ alone in the present value model of Equation (1), it is termed an
intrinsic rational bubble by Froot and Obstfeld (1991). Intrinsic bubbles turn out to be a
non-linear function of dividends. Their exact functional form depends on the assumed
stochastic process for the dividends.

3. SOLUTION TO THE MODEL

In this section, we obtain an exact analytical solution for the present value stock
price $P_t^{pv}$ when the dividend growth rate follows a random walk with drift with
innovations drawn from the family of stable distributions. The Gaussian random walk emerges as a special case. We also derive conditions under which a posited functional form for $B_t$ satisfies all the conditions for a rational intrinsic bubble.

3a. Specification of the Dividends process

We assume that log-dividends stochastically evolve according to the law of motion:

$$\ln(D_t) = \mu + \ln(D_{t-1}) + \xi_t, \quad \xi_t \sim \text{iid} S(\alpha, \beta, c, 0).$$

Here, $S(\alpha, \beta, c, 0)$ represents a stable distribution with characteristic exponent $\alpha$, skewness parameter $\beta$, scale parameter $c$, and location parameter set to zero. Appendix A defines these distributions and lists some of their properties.

For $s \geq t$, Equation (6) implies that:

$$D_s = D_t \exp\left[ (s - t)\mu + \xi_{t+1} + \xi_{t+2} + \ldots + \xi_{t+(s-t)} \right].$$

Substituting this into the solution for the fundamental stock price given in Equation (4) yields:

$$P^{PV}_t = D_t \sum_{s=t}^\infty e^{-r(s-t)+t} \mu \mathbb{E}_t \left[ \exp\left( \xi_{t+1} + \xi_{t+2} + \ldots + \xi_{t+(s-t)} \right) \right].$$

In deriving Equation (8), we assume that $D_t$ is contained in the information set available at the start of period $t$ on which the expectations $\mathbb{E}_t$ are based.
3b. Finiteness of Conditional Expectations

Given the iid nature of the innovations \( \{\xi_t\} \) to the dividends process, the expectations term on the right hand side of Equation (8) reduces to:

\[
E_t \left[ \exp(\xi_{t+1} + \xi_{t+2} + \ldots + \xi_{t+(s-t)}) \right] = E_t \left[ \exp(\xi_{t+1}) \right] E_t \left[ \exp(\xi_{t+2}) \right] \ldots E_t \left[ \exp(\xi_{t+(s-t)}) \right].
\]

(9)

When \( \{\xi_t\} \) is iid normal, each of the conditional expectations on the right hand side of the above equation are finite and are given by the moment generating function. However, when \( \{\xi_t\} \) is iid non-normal stable, i.e. when the exponent \( \alpha \) characterizing these innovations is less than 2, each of the conditional expectations is infinite, unless the skewness parameter \( \beta = -1 \) (see Appendix A). Equation (A8) gives the formulae for these conditional expectations.

3c. Solving for the Present Value Stock Price

Thus, under the assumption that dividends evolve according to the stochastic process given in Equation (6) with \( \beta = -1 \), one can now derive the present value stock price by evaluating the right hand side of Equation (8). The expression for \( P^v_t \) differs in the case when the characteristic exponent \( \alpha = 1 \) from that when \( \alpha \neq 1 \).\(^1\) In the rest of this

\(^1\)This arises because of two reasons. One reason is that the expressions for \( Ee^X \) differ in the two cases (see Equation (A8) in Appendix A). A second reason is that when we aggregate iid random variables with stable distributions, the expressions for the location
paper we focus our attention on the more general case \( \alpha \neq 1 \). All the results that follow for \( \alpha \neq 1 \) are also applicable for \( \alpha = 1 \) with appropriate modifications. The required derivations for \( \alpha = 1 \) do not pose any additional difficulties, and can be easily adapted from those given for \( \alpha \neq 1 \) in this paper.

Appendix B shows that the present value stock price is given by:

\[
P_{t}^{pv} = \kappa D_{t} \tag{10}
\]

where:

\[
\kappa = \left[ \frac{1}{\{\exp(r) - \exp(\mu - c^\alpha \sec(\pi \alpha / 2))\}} \right]. \tag{11}
\]

For convergence of the infinite summation in Equation (8), we need

\[ r > \mu - c^\alpha \sec(\pi \alpha / 2). \]

3d. Intrinsic Rational Bubbles

Let us postulate that intrinsic rational bubbles take the form given in Froot and Obstfeld (1991):

\[
B(D_t) = a_0 D_t^\lambda. \tag{12}
\]

Here, \( \lambda > 0 \) for the bubble to grow with an increase in dividends and \( a_0 > 0 \) to ensure non-negativity of stock prices.

parameter \( \delta \) for the aggregate random variable also differ in the two cases (see Equation (A7) in Appendix A).
Appendix C shows that the functional form for the intrinsic bubble in Equation (12) satisfies Equation (5) defining a bubble, provided that $\lambda$ is chosen to satisfy:

$$r = \lambda \mu - (\lambda c)^\alpha \sec(\pi \alpha / 2).$$

(13)

The inequality $r > \mu - c^\alpha \sec(\pi \alpha / 2)$ can be used to show that $\lambda > 1$ whenever the characteristic exponent $\alpha > 1$.

3e. Solution under Gaussian Random Walk

If the process for dividend growth rates is a Gaussian random walk plus drift, i.e. if the innovations in Equation (6) were Gaussian, then the solution for the present value stock price is easily obtained by setting $\alpha = 2$ in Equations (10) and (11) above. One can readily show that the expression obtained for the stock prices in this case is identical to the one given in Froot and Obstfeld (1991).

The conditions needed for convergence of the price-dividend ratio as well as the conditions for $B(D_t)$ to be a rational intrinsic bubble are also identical to those in Froot and Obstfeld (1991).

4. EMPIRICAL ASSESSMENT OF THE MODEL

4a. Characteristics of the Data

All data series used are taken from Shiller’s (1986) data appendix. The nominal stock prices are annual series of January values of the Standard and Poor Composite Stock Price Index (series 1 in Shiller’s dataset). The nominal dividend series are
dividends per share (series 2 in Shiller’s dataset). The producer price index is used as the
deflator to obtain real values (series 5 in Shiller’s dataset). This choice gives us the
longest sample length spanning the period 1900-1999. Although all three series are
available going back to 1871, we start the series in 1900 because Froot and Obstfeld
(1991) use data starting at this time point. They provide reasons for omitting data from
the earlier three decades.

Figure 1 plots real stock prices, real dividends, real dividend growth rates, and the
price-dividend ratios. Table 1 presents summary statistics on the dividend growth rates
and on the price-dividend ratios. A feature that emerges strongly from these statistics is
the leptokurtic nature of both series. Kurtosis is statistically significantly greater than
three indicating fat tails in the empirical histograms. Normality is strongly rejected for
both series. This provides the basis for our empirical specification that follows in the next
subsection.

4b. Econometric Specification

The empirical evaluation of the present value model requires specification of an
exogenous stochastic process for dividends. From Equation (6) and with the assumed
\( \beta = -1 \), we get:

\[
\ln(D_t) = \mu + \ln(D_{t-1}) + \xi_t, \quad \xi_t \sim \text{iid} \mathcal{S}(\alpha, -1, c, 0). \tag{14}
\]

From the discussion immediately following Equation (5), a complete solution to
the present value model can be written as:
\[ P_t = P_t^{pv} + B_t. \] (15)

This satisfies the present value model given by Equation (1) but violates the transversality condition given in Equation (3) for all \( B_t \neq 0 \). Using Equations (10), (12) and (13), one can write:

\[ P_t = \kappa D_t + a_0 D_t^\lambda. \] (16)

Motivated by this, one can then write down an econometric model for stock prices:

\[ P_t = b_0 D_t + b_1 D_t^{\lambda-1} + \varepsilon_t. \] (17)

This can be rewritten after dividing through by \( D_t \) as:

\[ \frac{P_t}{D_t} = b_0 + b_1 D_t^{\lambda-1} + \eta_t, \quad \eta_t \sim \text{iid} \mathcal{S}(\alpha, \eta_0, c_\eta, 0). \] (18)

where \( b_0, b_1, \lambda > 0 \). The error term \( \eta_t \) is assumed to be independent of the innovations \( \xi_t \), and of the dividends \( D_t \), at all leads and lags.

The empirical assessment of the present value model proceeds with estimation of Equations (14) and (18), subject to:

\[ r = \lambda \mu - (\lambda c_{\xi})^{\mu_x} \sec(\pi \alpha_x / 2). \] (19)

The null hypothesis of no bubbles implies that \( b_0 = \kappa \) and \( b_1 = 0 \), whereas the alternative hypothesis of a bubble implies that \( b_0 = \kappa \) and \( b_1 > 0 \).
4c. Random Walk Model Estimates for Real Dividends

Table 2 presents empirical results on maximum likelihood estimates of Equation (14). The first panel reports results on fitting a random walk with stable innovations to real dividends and the second panel reports results on fitting a Gaussian random walk. The characteristic exponent $\alpha$ is estimated to be 1.86, well below the bound of 2 that characterizes Gaussian distributions.

Following Froot and Obstfeld (1991), the constant discount factor is chosen to be $r = 0.086$. Using the estimates from maximum likelihood estimation of the random walk model, we verify that the convergence condition required to obtain the present value stock price in Equations (10 and (11) is satisfied. The model-implied price-dividend ratio $P^\text{pv}_t / D_t \equiv \kappa$ is reported in Table 3 to be 20.785. This agrees closely with the mean price-dividend ratio of 23.65 reported in the second row of Table 1. Solving the nonlinear Equation (19) yields $\lambda = 1.836$.

\footnote{Computing the probability densities for stable distributions poses a challenge. One way to evaluate these is by using Zolotarev’s (1986, p.74, 78) proper integral representations or by taking the inverse Fourier transform of their characteristic function given in Equations (A2) and (A3) in Appendix A. Here, we use the computational algorithm developed by J.P. Nolan (2000), archived at http://www.cas.american.edu/~jpnolan.}
From Table 3, with Gaussian innovations driving the random walk for dividends, the model-implied price-dividend ratio $\kappa$ is only 14.998, considerably below the empirically observed ratio. The exponent defining the bubble component $\lambda$ is higher at 2.487. For comparison we note that Froot and Obstfeld (1991), with a shorter and somewhat different data series, obtain an estimate of $\kappa = 14$ and $\lambda = 2.74$.

Thus, the stable model for dividends implies a constant theoretical price-dividend ratio that is close to the empirically observed mean. The Gaussian model also implies a constant theoretical price-dividend ratio but its value is low when compared to the empirically observed mean.

Also, the stable model gives a bubble component that is considerably less nonlinear (as measured by the value of the exponent $\lambda$) than that under the Gaussian model. This is in accord with the fact that accounting for fat tails reduces the degree of nonlinearity required to explain observed variation in price-dividend ratios (see, for instance, Bidarkota, 2000).

4d. Price-Dividend Ratio Regression Results

As noted at the end of subsection (4b), the empirical evaluation of the present value model could proceed by estimating Equations (14) and (18), subject to the restriction given in Equation (19). One could estimate all the parameters of the model jointly, by simultaneous estimation of the two equations. Or, alternatively, one could
estimate Equation (14) first, set $b_0 = \kappa$ and $\lambda$ equal to the value obtained by solving Equation (19), and then estimate Equation (18).

In what follows, we always estimate Equations (14) and (18) individually rather than simultaneously. The reason is technical. As noted in footnote 2, computing the probability densities for stable distributions poses a challenge. While the innovations to the log-dividends in Equation (14) have a skewness coefficient of –1, the error term in the price-dividend regression Equation (18) has a skewness coefficient of 0. We use McCulloch’s (1996b) GAUSS code for estimating the probability densities of the stable shocks in Equation (18), but this only works for errors that are symmetric. To estimate the random walk with maximally skewed stable errors in Equation (14), we use Nolan’s (2000) computer program available in digital Fortran (see footnote 2 for further details).

In our estimation of various versions of the price-dividend ratio regression that we report on below, we always set the exponent on the bubble term $\lambda$ at its value obtained by solving Equation (19). Froot and Obstfeld (1991) do estimate the price-dividend ratio regression this way and also alternatively by estimating $\lambda$ along with the other parameters of Equations (14) and (18) simultaneously. However, their inferences on the statistical significance of the bubble component in the two instances are qualitatively similar.

Finally, we estimate Equation (18) both by estimating $b_0$ as a free parameter and alternatively restricting $b_0 = \kappa$. We report on both results below.
Table 4 presents empirical results on maximum likelihood estimation of the nonlinear price-dividend regression given in (18). The first panel presents regression results with stable errors and a stable random walk process for dividends. Results are presented both for an unrestricted model in which the coefficient on the bubble component $b_1$ is estimated and a restricted model in which we set $b_1 = 0$. Further, within the unrestricted and restricted models, results are presented both for a version in which the intercept term $b_0$ is estimated as a free parameter and a restricted model in which we set $b_0 = \kappa$.

For the fully unrestricted model, we find from the first row that the characteristic exponent $\alpha$ is estimated to be 1.76, suggesting substantial fat tails compared to the Gaussian distribution. The intercept term $b_0$ is estimated to be 8.64. This is considerably lower than the theoretical price-dividend ratio $\kappa$ of 20.79. The likelihood ratio (LR) test indicates that the estimated $b_0$ is statistically significantly different from $\kappa$.

The coefficient on the bubble component $b_1$ is estimated to be 3.28. The LR test for $b_1 = 0$ is strongly rejected. On account of the explosive nature of the bubble term in Equation (18), Froot and Obstfeld (1991) show that the t-statistic for the hypothesis $b_1 = 0$ will have the normal distribution only if the regression residuals $\eta_t$ are normally distributed. In Equation (18), we have modeled these as being drawn from the stable distribution, however. In order to see whether our statistical inference on the existence of bubbles is affected by this assumption, we also estimated the price-dividend regression
equation with $\eta_t$ assumed normal, but the dividends process is still a random walk with stable innovations. Results are presented in the third panel of Table 4. As we can see, none of our inferences change qualitatively from those with stable regression errors.

Finally, panel 2 presents regression results with a Gaussian random walk process for dividends and Gaussian errors $\eta_t$ in the price-dividend regression. The estimated $b_0$ of 12.501 is now much closer in value to the theoretical price-dividend ratio $\kappa$ of 14.998 under a Gaussian random walk for dividends. Nonetheless, the hypothesis that $b_0 = \kappa$ is still strongly rejected. The coefficient on the bubble component $b_1$ is lower at only 0.732. Once again, one cannot reject the existence of bubbles. Thus, all our statistical inferences are qualitatively unchanged across all three panels of Table 4.

Figures 2-4 plot the observed price-dividend ratios and prices, along with the fitted values from the fully unrestricted models in panels 1-3 of Table 4, respectively. The contribution of the fundamental present value component and that of the bubble in accounting for observed variation in $P/D$ ratios and stock prices is clearly evident in the figures. There does not appear to be much of an improvement in overall fit of the model when one goes from Gaussian to stable distributions.

4e. Discussion of Results

The $P/D$ regression results reported in the previous subsection indicate that the hypothesis that $b_0 = \kappa$ is rejected across all three panels. This contrasts sharply with the
results in Froot and Obstfeld (1991). The difference is likely due to our longer data series containing observations from the bull market of the 1990s. However, the fact that we can reject the absence of bubbles across all three panels is in line with the inference in Froot and Obstfeld (1991). Thus, accounting for fat tails does not affect the qualitative outcome of testing this hypothesis.

The most significant difference between our results with and without fat tails is the estimate of the exponent on the bubble component $\lambda$. As reported in subsection (4c), the Gaussian random walk for log-dividends yields an estimate for $\lambda$ of 2.487 whereas the estimate implied by the stable random walk is only 1.836. The sensitivity of prices with respect to dividends, measured by $dP_t / dD_t$, works out to be 10.684 with the fully unrestricted stable $P/D$ regression results and 15.098 with the corresponding Gaussian regression. Thus, the stable model implies a lower sensitivity of prices to dividends.

5. CONCLUSIONS

We studied the present value model with a constant discount factor. The exogenous dividends are assumed to evolve as a random walk with innovations drawn from the family of stable innovations. We derived an analytical formula for the present value stock price in such a setting. Further extending the analysis in Froot and Obstfeld (1991) that developed a Gaussian framework, we derived a functional form for intrinsic bubble that violates the transversality condition.
We estimated the model with annual US stock price and dividends data over the last century. Our statistical rejection of the absence of a bubble component in annual US stock price data is unchanged when we account for fat tails in dividends and stock price data. However, accounting for fat tails leads to an intrinsic bubble component that is less non-linear, and consequently less explosive, than in the Gaussian setup. This setup also yields lower sensitivity of prices to changes in dividends than is implied by the Gaussian framework.
APPENDIX A
Stable Distributions and Their Properties

This section draws heavily from McCulloch (1996a). Stable distributions S(x; \alpha, \beta, c, \delta) are determined by four parameters. The location parameter \delta \in (-\infty, \infty) shifts the distribution to the left or right, while the scale parameter c \in (0, \infty) expands or contracts it about \delta, so that

\[ S(x; \alpha, \beta, c, \delta) = S((x - \delta) / c; \alpha, \beta, 1, 0). \] (A1)

The standard stable distribution function has c = 1 and \delta = 0. If a random variable X has a stable distribution, it is represented as X \sim S(\alpha, \beta, c, \delta).

The characteristic exponent \alpha \in (0, 2] governs the tail behavior, and therefore the degree of leptokurtosis. When \alpha = 2, the normal distribution results, with variance 2c. For \alpha < 2, the variance is infinite. When \alpha > 1, E(X) = \delta; but if \alpha \leq 1, the mean is undefined.

The skewness parameter \beta \in [-1, 1] is defined such that \beta > 0 indicates positive skewness. If \beta = 0, the distribution is symmetric stable. As \alpha \uparrow 2, \beta loses its effect and becomes unidentified.

Stable distributions are defined most concisely in terms of their log-characteristic functions:

\[ \ln \mathbb{E} \exp(itX) = i\delta t + \psi_{\alpha,\beta}(ct) \] (A2)

where

\[ \psi_{\alpha,\beta}(t) = \begin{cases} -|t|^\alpha (1-i\beta \text{sign}(t) \tan(\pi \alpha / 2)) & \text{for } \alpha \neq 1 \\ -|t|(1+i\beta (2/\pi) \text{sign}(t) \ln |t|) & \text{for } \alpha = 1 \end{cases} \] (A3)

is the log-characteristic function for S(\alpha, \beta, 1, 0).
When $\alpha < 2$, stable distributions have tails that behave asymptotically like $x^{-\alpha}$ and give the stable distributions infinite absolute population moments of order greater than or equal to $\alpha$.

Let $X \sim S(\alpha, \beta, c, \delta)$ and $a$ be any real constant. Then (A2) implies:

$$aX \sim S(\alpha, \text{sign}(a)\beta, |a|c, a\delta).$$

(A4)

Let $X_1 \sim (\alpha, \beta_1, c_1, \delta_1)$ and $X_2 \sim (\alpha, \beta_2, c_2, \delta_2)$ be independent drawings from stable distributions with a common $\alpha$. Then $Y = X_1 + X_2 \sim S(\alpha, \beta, c, \delta)$, where

$$c^\alpha = c_1^\alpha + c_2^\alpha$$

(A5)

$$\beta = (\beta_1 c_1^{\alpha} + \beta_2 c_2^{\alpha}) / c^\alpha$$

(A6)

$$\delta = \begin{cases} 
\delta_1 + \delta_2 & \text{for } \alpha \neq 1 \\
\delta_1 + \delta_2 + 2(\beta c \ln(c) - \beta_1 c_1 \ln(c_1) - \beta_2 c_2 \ln(c_2)) / \pi & \text{for } \alpha = 1.
\end{cases}$$

(A7)

When $\beta_1 = \beta_2$, $\beta$ equals their common value, so that $Y$ has the same shaped distribution as $X_1$ and $X_2$. This is the “stability” property of stable distributions that leads directly to their role in the central limit theorem, and makes them particularly useful in financial portfolio theory. When $\beta_1 \neq \beta_2$, $\beta$ lies between $\beta_1$ and $\beta_2$.

For $\alpha < 2$ and $\beta > -1$, the long upper Paretian tail of $X \sim S(\alpha, \beta, c, \delta)$ makes $\text{Ee}^X$ infinite. However, when $\beta = -1$,

$$\ln \text{Ee}^X = \begin{cases} 
\delta - c^\alpha \sec(\pi \alpha / 2), & \alpha \neq 1 \\
\delta + (2c / \pi) \ln c, & \alpha = 1
\end{cases}$$

(A8)

This formula greatly facilitates asset pricing under log-stable uncertainty.

See also Zolotarev (1986, p.112) and McCulloch (1996a).
APPENDIX B

Derivation of the Present Value Stock Price

In this appendix we derive the solution for the present value stock price given by Equations (10) and (11). As noted in the first paragraph of subsection (3c), we only derive the formula for the stock price in the case $\alpha \neq 1$.

From Equation (8),

$$P_t^{pv} = D_t \sum_{s=t}^{\infty} e^{-r(s-t+1)+(s-t)\mu} E_t \left[ \exp\left(\xi_{t+1} + \xi_{t+2} + \ldots + \xi_{t+(s-t)}\right) \right]. \quad (B1)$$

Substituting Equation (9) into the above equation yields:

$$P_t^{pv} = D_t \sum_{s=t}^{\infty} e^{-r(s-t+1)+(s-t)\mu} E_t \left[ \exp\left(\xi_{t+1}\right)\exp\left(\xi_{t+2}\right) \ldots \exp\left(\xi_{t+(s-t)}\right) \right]. \quad (B2)$$

From Equation (6), $\xi_t \sim \text{iid } S(\alpha, \beta, c, 0)$. With $\beta = -1$ assumed in the derivation of Equation (10) and using Equation (A8) in Appendix A, we get:

$$E_t \left[ \exp(\xi_{t+1}) \right] = E_t \left[ \exp(\xi_{t+2}) \right] = \ldots = E_t \left[ \exp(\xi_{t+(s-t)}) \right] = \exp(-c^\alpha \sec(\pi \alpha / 2)). \quad (B3)$$

Substituting Equation (B3) into Equation (B2) yields:

$$P_t^{pv} = D_t \sum_{s=t}^{\infty} e^{-r(s-t+1)+(s-t)\mu} \left[ \exp\left(-c^\alpha \sec(\pi \alpha / 2)\right) \right]^{s-t}. \quad (B4)$$

This can be rewritten as:

$$P_t^{pv} = D_t e^{-r} \left[ 1 + \sum_{s=t+1}^{\infty} \exp(s-t)\left(-r + \mu - c^\alpha \sec(\pi \alpha / 2)\right) \right]. \quad (B5)$$
The infinite summation in the above equation converges only if \( r > \mu - e^\alpha \sec(\pi \alpha / 2) \). In this case, from the sum of an infinite geometric progression, we find:

\[
P_t^{pv} = \left[ \frac{1}{\exp(r)} - \exp\left(\mu - e^\alpha \sec(\pi \alpha / 2)\right) \right] D_t. \tag{B6}
\]
or

\[
P_t^{pv} = \kappa D_t \tag{B7}
\]

where:

\[
\kappa = \left[ \frac{1}{\exp(r)} - \exp\left(\mu - e^\alpha \sec(\pi \alpha / 2)\right) \right]. \tag{B8}
\]

APPENDIX C

**Intrinsic Bubbles under Stable Random Walk plus Drift**

In this appendix we demonstrate that \( B_t \) given by Equation (12) is an intrinsic rational bubble when log-dividends evolves as a stable random walk plus drift as given in Equation (6).

From Equation (12),

\[
B(D_t) = a_0 D_t^\lambda, \quad a_0 > 0. \tag{C1}
\]

Now, \( B(D_t) \) is a rational intrinsic bubble if it satisfies Equation (5), which is given as:

\[
B_t = e^{-r} E_t \{ B_{t+1} \}. \tag{C2}
\]

Equation (6) implies that:

\[
D_{t+1} = D_t \exp[\mu + \xi_{t+1}]. \tag{C3}
\]
Therefore,

\[ D_{t+1}^\lambda = D_t^\lambda \exp[\lambda \mu + \lambda \xi_{t+1}]. \] \hspace{1cm} (C4)

From Equation (6), \( \xi_t \sim \text{iid } S(\alpha, \beta, c, 0) \). With \( \beta = -1 \) and \( \lambda > 0 \), Equation (A4) from Appendix A yields \( \xi_t \sim \text{iid } S(\alpha, -1, \lambda c, 0) \). Using Equation (A8) in Appendix A, we get:

\[ E_t \{ \exp(\lambda \xi_{t+1}) \} = \exp \left[ - (\lambda c)^\alpha \sec(\pi \alpha / 2) \right]. \] \hspace{1cm} (C5)

Now, using Equation (C1) one can write the right hand side of Equation (C2) as:

\[ e^{-r} E_t \{ B_{t+1} \} = e^{-r} E_t \left\{ a_0 D_{t+1}^\lambda \right\}. \] \hspace{1cm} (C6)

Substituting Equation (C4) into (Equation (C6) yields:

\[ e^{-r} E_t \{ B_{t+1} \} = a_0 e^{-r} D_t^\lambda E_t \{ \exp[\lambda \mu + \lambda \xi_{t+1}] \}. \] \hspace{1cm} (C7)

Now, substituting Equation (C5) into Equation (C7) gives:

\[ e^{-r} E_t \{ B_{t+1} \} = a_0 D_t^\lambda \exp \left[ - r + \lambda \mu - (\lambda c)^\alpha \sec(\pi \alpha / 2) \right]. \] \hspace{1cm} (C8)

Thus, Equation (C2) is satisfied, provided that:

\[ r = \lambda \mu - (\lambda c)^\alpha \sec(\pi \alpha / 2). \] \hspace{1cm} (C9)
### Table 1: Summary Statistics of the Data

<table>
<thead>
<tr>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Normality test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.500e-2</td>
<td>1.590e-2</td>
<td>-0.579</td>
<td>6.055</td>
<td>44.01</td>
</tr>
<tr>
<td>(1.26e-2)</td>
<td>(2.24e-3)</td>
<td>(0.991)</td>
<td>(2.75e-10)</td>
<td>(2.77e-10)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Price-Dividend ratios</th>
<th>23.650</th>
<th>84.290</th>
<th>2.570</th>
<th>13.083</th>
<th>533.65</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0.92)</td>
<td>(11.92)</td>
<td>(4.77e-26)</td>
<td>(2.01e-94)</td>
<td>(1.32e-116)</td>
</tr>
</tbody>
</table>

**Notes to Table 1:**

1. Numbers in parentheses in the first two columns are the standard errors for the mean and variance.

2. Numbers in parentheses in the third and fourth columns are the p-values for the null hypothesis of no skewness and no excess kurtosis, respectively.

3. The normality test gives the Jarque-Bera test statistic and the p-value in parentheses.
### Table 2: Maximum Likelihood Model Estimates

\[
\ln(D_t) = \mu + \ln(D_{t-1}) + \xi_t, \quad \xi_t \sim \text{iid } S(\alpha_{\xi},-1,c_{\xi},0). \quad (14)
\]

#### Panel 1: Stable Random Walk

<table>
<thead>
<tr>
<th>$\alpha_{\xi}$</th>
<th>$c_{\xi}$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.859</td>
<td>0.077</td>
<td>0.032</td>
</tr>
<tr>
<td>(0.194)</td>
<td>(0.012)</td>
<td>(0.026)</td>
</tr>
</tbody>
</table>

#### Panel 2: Gaussian Random Walk

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sigma^2_{\xi}$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (restricted)</td>
<td>0.016</td>
<td>0.015</td>
</tr>
<tr>
<td>(2.24e-3)</td>
<td>(1.26e-2)</td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table 2:

1. When $\alpha = 2$, errors are Gaussian with variance $\sigma^2 = 2c^2$.
2. Numbers in parentheses for panel 1 are the 95 percent confidence interval estimates.
3. Numbers in parentheses for panel 2 are the standard errors.
Table 3: Implied Parameter Values

<table>
<thead>
<tr>
<th></th>
<th>Discount factor</th>
<th>$\lambda$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable random walk for Dividends</td>
<td>0.086</td>
<td>1.836</td>
<td>20.785</td>
</tr>
<tr>
<td>Gaussian random walk for Dividends</td>
<td>0.086</td>
<td>2.487</td>
<td>14.998</td>
</tr>
</tbody>
</table>
Table 4: Price-Dividend Ratio Regression Estimates

Panel 1: Stable Random Walk plus Stable Errors
\[
\frac{P_t}{D_t} = b_0 + b_1 D_t^{\lambda - 1} + \eta_t, \quad \eta_t \sim \text{iid } S(\alpha_\eta, 0, c_\eta, 0). \quad (18)
\]

<table>
<thead>
<tr>
<th></th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(\alpha_\eta)</th>
<th>(c_\eta)</th>
<th>log L</th>
<th>(2\Delta \log L) for (b_0 = \kappa)</th>
<th>(2\Delta \log L) for (b_1 = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unrestricted</strong> model</td>
<td>8.637</td>
<td>3.281</td>
<td>1.759</td>
<td>3.041</td>
<td>-303.86</td>
<td>61.30</td>
<td>79.52</td>
</tr>
<tr>
<td>(restricted to (\kappa))</td>
<td>(1.359)</td>
<td>(0.306)</td>
<td>(0.097)</td>
<td>(0.260)</td>
<td>(4.9e-15)</td>
<td>(4.8e-19)</td>
<td></td>
</tr>
<tr>
<td><strong>Restricted</strong> model</td>
<td>22.235</td>
<td>1.730</td>
<td>4.428</td>
<td>-343.62</td>
<td>4.34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(restricted to (\kappa))</td>
<td>(0.694)</td>
<td>(0.132)</td>
<td>(0.385)</td>
<td>(0.04)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20.785</td>
<td>1.624</td>
<td>4.334</td>
<td>-345.79</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(restricted to (\kappa))</td>
<td>(0.182)</td>
<td>(0.441)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table 3:

1. Unrestricted model is one in which \(b_1\) is estimated. Restricted model sets \(b_1 = 0\).

2. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.

3. \(2\Delta \log L\) gives the likelihood ratio (LR) test statistics. P-values from the \(\chi^2_1\) distribution are in parentheses.
Panel 2: Gaussian Random Walk plus Gaussian Errors

\[
P_t \frac{D_t}{D_t} = b_0 + b_1 D_t^{\lambda-1} + \eta_t, \quad \eta_t \sim \text{iid } N(0, \sigma_{\eta}^2).
\]

<table>
<thead>
<tr>
<th></th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(\sigma_{\eta}^2)</th>
<th>(\log L)</th>
<th>(2\Delta \log L) for (b_0 = \kappa)</th>
<th>(2\Delta \log L) for (b_1 = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unrestricted</strong></td>
<td>12.501</td>
<td>0.732</td>
<td>29.297</td>
<td>-</td>
<td>6.32</td>
<td>105.68</td>
</tr>
<tr>
<td>model</td>
<td>(0.977)</td>
<td>(0.053)</td>
<td>(4.143)</td>
<td>310.77</td>
<td>(0.01)</td>
<td>(8.7e-25)</td>
</tr>
<tr>
<td></td>
<td>14.998</td>
<td>0.618</td>
<td>31.209</td>
<td>-</td>
<td></td>
<td>162.90</td>
</tr>
<tr>
<td>(restricted to (\kappa))</td>
<td>(0.031)</td>
<td>(4.414)</td>
<td>313.93</td>
<td></td>
<td></td>
<td>(2.6e-37)</td>
</tr>
<tr>
<td><strong>Restricted</strong></td>
<td>23.650</td>
<td>84.290</td>
<td></td>
<td>-</td>
<td>63.54</td>
<td></td>
</tr>
<tr>
<td>model</td>
<td>(0.918)</td>
<td>(11.920)</td>
<td>363.61</td>
<td></td>
<td>(1.6e-15)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14.998</td>
<td>159.145</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(restricted to (\kappa))</td>
<td>(22.506)</td>
<td>395.38</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table 3:

1. Unrestricted model is one in which \(b_1\) is estimated. Restricted model sets \(b_1 = 0\).
2. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.
3. \(2\Delta \log L\) gives the likelihood ratio (LR) test statistics. P-values from the \(\chi_1^2\) distribution are in parentheses.
Panel 3: Stable Random Walk plus Gaussian Errors

\[ \frac{P_t}{D_t} = b_0 + b_1 D_{t-1} + \eta_t, \quad \eta_t \sim \text{iid } N(0, \sigma_\eta^2). \]

<table>
<thead>
<tr>
<th>b_0</th>
<th>b_1</th>
<th>\sigma_\eta^2</th>
<th>\log L</th>
<th>2\Delta \log L \text{ for } b_0 = \kappa</th>
<th>2\Delta \log L \text{ for } b_1 = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unrestricted model</td>
<td>4.794</td>
<td>4.297</td>
<td>35.191</td>
<td>-319.93</td>
<td>63.22</td>
</tr>
<tr>
<td></td>
<td>(1.703)</td>
<td>(0.364)</td>
<td>(4.977)</td>
<td>(1.8e-15)</td>
<td>(9.0e-21)</td>
</tr>
<tr>
<td></td>
<td>20.785</td>
<td>1.095</td>
<td>66.220</td>
<td>-351.54</td>
<td>33.42</td>
</tr>
<tr>
<td></td>
<td>(restricted to ( \kappa ))</td>
<td>(0.174)</td>
<td>(9.365)</td>
<td></td>
<td>(7.4e-9)</td>
</tr>
<tr>
<td>Restricted model</td>
<td>23.650</td>
<td>84.290</td>
<td>-363.61</td>
<td>9.28</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.918)</td>
<td>(11.920)</td>
<td></td>
<td>(2.3e-3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20.785</td>
<td>92.495</td>
<td>-368.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(restricted to ( \kappa ))</td>
<td>(13.081)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Notes to Table 3:

1. Unrestricted model is one in which \( b_1 \) is estimated. Restricted model sets \( b_1 = 0 \).

2. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.

3. \( 2\Delta \log L \) gives the likelihood ratio (LR) test statistics. P-values from the \( \chi^2_1 \) distribution are in parentheses.
Figure 1. Plots of Raw Data

**Fig 1a. Real Stock Prices**

**Fig 1b. Real Dividends**

**Fig 1c. Growth Rates of Real Dividends**

**Fig 1d. Price-Dividend Ratios**
Figure 2. Results with Stable Random Walk and
Stable Price-Dividend Ratio Regression

Fig 2a. Observed Price-Dividend Ratios and Fitted Values

Fig 2b. Observed Stock Prices and Fitted Values
Figure 3. Results with Gaussian Random Walk and
Gaussian Price-Dividend Ratio Regression
Figure 4. Results with Stable Random Walk and Gaussian Price-Dividend Ratio Regression
REFERENCES


