On the Economic Impact of Modeling Non-Linearities: 
The Asset Pricing Example

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Abstract: We investigate the economic importance of modeling non-linearities in the dynamics of exogenous processes on the implied moments of endogenous variables in the context of the consumption-based asset pricing model. For this purpose, we model the endowment process alternatively as a linear autoregression and as a non-linear threshold autoregression. The asset pricing model with non-linear endowment is solved using quadrature techniques. A comparison of the moments of the model-implied rates of return in the two cases suggests that the economic impact of modeling non-linearities is less than 0.01 percent per annum.

Key phrases: asset pricing; rates of returns; non-linearities; threshold autoregressions; numerical solutions

JEL Codes: G12, C22, C52, C63

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Introduction

A number of studies have documented statistically significant non-linearities in the dynamics of several macroeconomic time series. For instance, within a univariate setting, non-linearities in U.S. aggregate income series have been reported in Neftci (1984), Hamilton (1989), Potter (1995), Bidarkota (2000), and several others. Although such non-linearities have been extensively documented, nonetheless most macroeconomic studies utilize simple, invariably linear, stochastic processes for characterizing exogenous variables in the models. For instance, much of the real business cycle literature uses simple autoregressive technology processes for driving economic fluctuations. Under special circumstances, such simplicity affords tractable exact analytical solutions to the endogenous variables of the model.

On the other hand, non-linear forcing processes typically prelude the possibility of finding analytical solutions in most settings. An important exception in this context is the Markov switching model due to Hamilton (1989). This non-linear model has been widely used to characterize the exogenous driving processes, especially in the asset pricing literature, on account of the analytical tractability it affords (Cecchetti et al. 1993, Bonomo and Garcia 1994).

In this paper, we ask what is the economic cost of ignoring non-linearities in macro models? We address this question in the context of the consumption-based asset pricing model of Lucas (1978). We solve the model under two alternative assumptions on the exogenous dividend process. In the benchmark case, dividends are modeled as a simple autoregression (AR). In the non-linear case, dividends are modeled as threshold autoregressive processes (Tong and Lim, 1980). In the benchmark case, an exact solution
to the model is available (Burnside 1998). In the non-linear case, an exact analytical solution is not tractable. In this case, we solve the model numerically, by first using Markov chain approximations for the dividends process (Tauchen, 1986) and subsequently solving the Euler equations of the model by quadrature techniques (Tauchen and Hussey, 1991). Moments of the model-implied endogenous variables, including rates of return, are then compared across the two cases to evaluate the economic impact of modeling non-linearities.

A few studies on asset pricing have utilized non-linear driving processes. For instance, Kandel and Stambaugh (1990), Tauchen and Hussey (1991), and more recently Ebell (2001) use an AR process with autoregressive conditional heteroskedasticity (ARCH) to model dividends. However, while the focus of the work in Tauchen and Hussey is to illustrate the quadrature solution technique, the focus of the work in the other two studies is on drawing out the implications of ARCH on the conditional moments of asset return dynamics.

A systematic evaluation of the effects of non-linearities on the economic implications of macro models has so far not been undertaken. We use the standard asset-pricing model here to analyze the economic impact of non-linearities because of the simplicity that the model affords (it has only one state variable) and because exact analytical solutions to the equilibrium quantities of interest are available at least in the benchmark linear endowment case. Our efforts are not to be viewed as a test of this asset-pricing model. The failure of the standard version of this model in replicating the
empirical features of observed data on equity and bond returns has been thoroughly established.¹

This paper is organized as follows. In section 2, we briefly sketch the asset pricing model, outline its solution under a linear endowment process, and describe the solution under a non-linear endowment process. In section 3, we evaluate the accuracy of discrete-valued Markov chains in approximating continuous-valued linear and non-linear stochastic processes with a simulation study. In section 4, we undertake an empirical analysis of the asset pricing model with linear endowment, and evaluate the accuracy of the numerical solution to the asset pricing model. In section 5, we extend the empirical analysis to a non-linear endowment process, and evaluate the effects of non-linearities. In section 6, we evaluate the impact of non-linearities on model-implied rates of return. In the final section, we conclude with a summary of the implications of non-linearities emerging from our study.

2. The Model and Its Solution

Sub-section 2.1 sketches the asset pricing model, sub-section 2.2 outlines the exact solution to the model with a linear autoregressive endowment process, and sub-section 2.3 describes a numerical solution to the model with a non-linear threshold autoregressive endowment process.

¹ See Kocherlakota (1996) for a survey of this literature.
2.1 The Asset Pricing Model

In a single good Lucas (1978) economy, with a representative maximizing agent and a single asset that pays exogenous dividends of non-storable consumption goods, the first-order Euler condition is:

\[ P_t U'(C_t) = \Theta E_t U'(C_{t+1})[P_{t+1} + D_{t+1}] \]  

(1)

Here, \( P_t \) is the real price of the single asset in terms of the consumption good \( U'(C) \) is the marginal utility of consumption \( C \) for the representative agent \( \Theta \) is a subjective discount factor, assumed non-stochastic and constant \( D \) is the dividend from the single productive unit \( E_t \) is the mathematical expectation, conditioned on information available at time \( t \).

Assume a constant relative risk aversion (CRRA) utility function:

\[ U(C) = (1 - \gamma)^{-1} C^{(1-\gamma)}, \quad \gamma \geq 0. \]  

(2)

Since consumption simply equals dividends in this simple model, i.e. \( C = D \) every period, Equation (1) reduces to:

\[ P_tD_t^{-\gamma} = E_t \Theta D_{t+1}^{-\gamma}[P_{t+1} + D_{t+1}] \]  

(3)

On rearranging, this yields:

\[ P_t = E_t \Theta \left( \frac{D_{t+1}}{D_t} \right)^{-\gamma} [P_{t+1} + D_{t+1}] \]  

(4)
Let \( v_t \) denote the price-dividend ratio, i.e. \( v_t = \frac{P_t}{D_t} \). Then, we can rewrite Equation (4) in terms of \( v_t \) as:

\[
v_t = E_t \left( \frac{D_{t+1}}{D_t} \right)^{1-\gamma} [v_{t+1} + 1]. \tag{5}
\]

Thus, this equation implicitly defines the solution to the asset pricing problem in this model. One specifies an exogenous stochastic process for dividends and solves for the price dividend ratio \( v_t \).

Using \( y_t = \ln(\frac{D_t}{D_{t-1}}) \) to denote the dividend growth rate, we can express Equation (5) as:

\[
v_t = E_t \theta \exp[(1-\gamma)y_{t+1}] (v_{t+1} + 1). \tag{6}
\]

Defining \( m_{t+1} \equiv \theta \exp[(1-\gamma)y_{t+1}] \), we can rewrite Equation (6) as:

\[
v_t = E_t m_{t+1} [v_{t+1} + 1]. \tag{7}
\]

On forward iteration, this equation yields:

\[
v_t = \sum_{i=1}^{\infty} E_t \left( \prod_{j=1}^{i} m_{t+j} \right) + \lim_{i \to \infty} E_t \prod_{j=1}^{i} m_{t+j} v_{t+i}. \tag{8}
\]

One solution to the above difference equation in \( v_t \) is obtained by imposing the transversality condition:

\[
\lim_{i \to \infty} \left( E_t \prod_{j=1}^{i} m_{t+j} v_{t+i} \right) = 0. \tag{9}
\]
This condition rules out solutions to the asset pricing model that imply intrinsic bubbles (Froot and Obstfeld, 1991). Imposing the transversality condition on Equation (8) gives:

\[
\nu_t = \sum_{i=1}^{\infty} E_t \left( \prod_{j=1}^{i} m_{t+j} \right). \tag{10}
\]

### 2.2 Solution under AR(1) Dividend Growth Rates

As a benchmark case, we assume that the dividend growth rates stochastically evolve according to:

\[
y_t = \mu + \rho (y_{t-1} - \mu) + \eta_t, \quad \eta_t \sim \text{iid } N(0, \sigma_{\eta}^2). \tag{11}
\]

Here, \( |\rho| < 1 \) to impose stationarity of the dividends process.

Under such an AR(1) endowment process, Burnside (1998) derives an exact analytical solution to the price-dividend ratio in the asset pricing model. His solution is given by:

\[
\nu = \sum_{i=1}^{\infty} \theta^i \exp \left[ a_i + b_i (y_t - \mu) \right]. \tag{12}
\]

In this solution, the constants \( a_i \) and \( b_i \) are given by:

\[
a_i = \gamma (1-\gamma) \mu + \frac{1}{2} (1-\gamma)^2 \left( \frac{\sigma^2}{(1-\rho)^2} \left[ i - 2 \frac{\rho}{(1-\rho)^i} \right]\right), \tag{13}
\]

\[
b_i = (1-\gamma) \frac{\rho}{(1-\rho)} \left( 1 - \rho^i \right). \tag{14}
\]

Burnside (1998) shows that the infinite series in Equation (12) converges if and only if

\[
\theta \exp \left[ (1-\gamma) \mu + \frac{1}{2} (1-\gamma)^2 \frac{\sigma^2}{(1-\rho)^2} \right] < 1.
\]
2.3 Solution under SETAR(1,1) Dividend Growth Rates

To assess the impact of non-linearities, we assume that the dividend growth rates stochastically evolve as a self-exciting threshold autoregressive (SETAR) process. Threshold autoregressions were introduced in Tong and Lim (1980), and versions of these models were fit to U.S. GNP series by Potter (1995), Bidarkota (2000), and others.

A SETAR(1,1) process can be written as follows. In regime 1, the process evolves as the following first-order autoregression:

\[ y_t = \mu_1 + \rho_1 (y_{t-1} - \mu_1) + \eta_{1t}, \quad \eta_{1t} \sim \text{iid } N(0, \sigma_{\eta_1}^2). \] (15.1)

In regime 2, the process evolves as the following alternative first-order autoregression:

\[ y_t = \mu_2 + \rho_2 (y_{t-1} - \mu_2) + \eta_{2t}, \quad \eta_{2t} \sim \text{iid } N(0, \sigma_{\eta_2}^2). \] (15.2)

In order to facilitate comparison with the linear AR(1) process and a statistical test for linearity later, we use first-order autoregressions in the two regimes.

The switch between the two regimes is governed by the past values of the variable \( y_t \). This feature makes these processes self-exciting. For instance, the switch could be governed by \( y_{t-1} > s \), and \( l \) and \( s \) are estimated along with other parameters of the model. Potter (1995) reports estimates of the delay parameter \( l \) and the threshold parameter \( s \) of 2 and 0 respectively for quarterly U.S. GNP series. As discussed by him, these estimates closely match those obtained by other scholars. We therefore simply set \( l = 2 \) and \( s = 0 \) in our empirical work that follows. Thus, we get regime 1 whenever \( y_{t-2} > 0 \), and we get regime 2 whenever \( y_{t-2} \leq 0 \).

Exact analytical solutions to the Lucas (1978) asset pricing model are available only in a handful of special cases, typically with a simple endowment process (Bidarkota...
and McCulloch 2003, Cecchetti et al. 1993, Tsionas 2003). When an exact analytical solution is not tractable, we need to resort to numerical techniques for finding approximate solutions to the model-implied price-dividend ratio. This involves solving the integral Equation (5). Tauchen and Hussey (1991) provide one method for finding an approximate solution to Equation (5). Intuitively, their method is simple and very appealing. The method involves first approximating the (typically) continuous-valued state vector in the model by a discrete-valued Markov chain. Once the Markov chain approximation is found, the integral asset pricing Equation (5) reduces to a system of linear simultaneous equations. The solution to these equations gives the price-dividend ratios at the discretized state space of the approximating Markov chain. The solution can then be extended to the entire continuous-valued domain of the true state vector using techniques such as Nystrom’s extension (see Tauchen and Hussey, 1991).

One can gauge the accuracy of the approximate solution by comparing it to the exact solution available under special cases. For instance, we can solve the asset pricing model for the price-dividend ratio when the dividend growth rates evolve as a first-order autoregression using Burnside’s (1998) exact solution and Tauchen-Hussey’s (1991) approximate solution, and compare the two to get a sense of the approximation errors.

3. Approximating Continuous-Valued Stochastic Processes with Discrete-Valued Markov Chains

In this section, we provide a measure of the accuracy involved in approximating a continuous-valued stochastic process with a discrete-valued Markov chain. We do this for
two different stochastic processes. One is a linear first-order Gaussian autoregression and the second is a non-linear conditionally Gaussian SETAR(1,1) process.

In order to assess the accuracy of the Markov chain approximation technique, we conducted the following two exercises. First, we simulated a simple first-order Gaussian autoregressive process and obtained a sample of 5000 observations. We estimated an AR(1) process by maximum likelihood (ML) with this simulated data. We then fit a Markov chain approximation to the simulated AR process using Tauchen’s (1986) approximation method, using 75 nodes. Subsequently, we obtained a simulated sample of 5000 observations from this approximating Markov chain. We then estimated an AR(1) process by maximum likelihood with this simulated data. The ML estimates of the AR(1) process obtained with the two simulated samples are reported in Table 1, along with the true parameters values. The table shows that the Markov chain approximates an AR(1) process quite well.

In the second exercise, we repeated the above with a Gaussian SETAR(1,1) process. The switch is governed by the value of the variable two periods earlier. If it is positive, the process is in regime 1, otherwise the process is in regime 2. We first simulated 5000 observations from such a SETAR(1,1) process, and estimated a similar SETAR(1,1) model by ML with the simulated data. We then obtained a Markov chain approximation to the SETAR(1,1) process, simulated 5000 observations from this approximating process, and then fit a SETAR(1,1) model by ML to this data. The ML estimates of the SETAR(1,1) process fit to the two simulated samples are presented in Table 2, along with the true parameter values. Once again, this table shows that the ML estimates from the two samples are quite close to their true parameter values.
Thus, the Markov chain with 75 nodes seems to approximate both a linear AR(1) and a non-linear SETAR(1,1) process quite well.

4. Asset Pricing with Linear AR(1) Endowment

In this section, we first describe the data used to proxy for the endowment in the model, and provide maximum likelihood estimates obtained by fitting an AR(1) process to this data series. We then evaluate the model-implied price-dividend ratios obtained from the exact solution to the model under the AR(1) endowment process. We go on to compare these price-dividend ratios with those obtained by solving the model numerically, using the techniques in Tauchen and Hussey (1991), in order to get a sense of the accuracy of the numerical solution method.

In the standard Lucas (1978) model, since consumption, dividends, and output are identical, we could estimate our endowment process with observed data on any of these three variables as proxies for the endowment in the model. In this study we use GNP. This is primarily because several studies including Potter (1995) and Bidarkota (2000) have fit SETAR processes to this version of the aggregate income series. We use quarterly US real GNP data obtained from the Survey of Current Business. The dataset spans the period 1947:1-1996:4. Figure 1 plots the real GNP growth rates.

Table 1 presents ML estimates of a simple first-order Gaussian autoregression fit to the GNP growth rates. The mean growth rate is estimated to be 0.008 per quarter (or, 3.2 percent per annum). The persistence is quite strong, with the AR(1) parameter estimated at 0.36 and strongly statistically significant. The second-order partial
The autocorrelation coefficient of the growth rates is 0.07. Thus, an AR(1) process seems adequate for capturing the bulk of the persistence in the growth rates.

Figure 2a shows the model-implied price-dividend (P/D) ratios obtained from the asset pricing model with a linear AR(1) endowment process. The P/D ratios were computed using the exact solution given by Burnside (1998), reproduced in Equation (12). The infinite summation was truncated to the first 10,000 terms. We verified that the P/D ratios were indistinguishable at truncations of 1,000 and 10,000 terms, indicating that the infinite summation had converged by the first 1,000 terms.

The mean P/D ratio is about 28.55, and it varies within a range of only about 0.06 of its mean value. This inability of the consumption-based asset pricing model to generate sufficient variation in the implied P/D ratios is well known in the literature.

Figure 2b shows the P/D ratios computed using the approximate solution method of Tauchen and Hussey (1991), after fitting an approximate Markov chain to the linear AR(1) endowment process using techniques in Tauchen (1986). Comparing the approximate P/D ratios to their exact values in Figure 2a suggests that the approximation is generally quite good. The exact and approximate P/D ratios plotted in Figure 2c are virtually indistinguishable. Their differences, exact minus approximate P/D ratios, plotted in Figure 2d show a maximum approximation error of only 0.003. Compared to the mean P/D ratio of 28.55, the approximation error is quite small.

5. Asset Pricing with Non-Linear SETAR(1,1) Endowment

In this section, we first provide ML estimates obtained by fitting a SETAR(1,1) process to the endowment data. We then evaluate the model-implied P/D ratios in the
asset pricing model driven by the estimated SETAR(1,1) endowment process. Finally, we compare these P/D ratios with those obtained with the linear endowment process discussed in section 4 earlier.

Table 4 presents ML estimates of a SETAR(1,1) process fit to the GNP growth rates. The difference in the mean growth rates in the two regimes is only 0.0008 per quarter (or, 0.32 percent per annum). The second regime (corresponding to lower growth rates) is more volatile but less persistent than the first regime (corresponding to higher growth rates). Overall, the non-linear effects are quantitatively weak. This, of course, has an important bearing on the overall impact of non-linearities, as we shall see subsequently.

The null hypothesis of a single regime (the null of linearity) can be tested by testing for equality of the parameter values in the two regimes. In general SETAR models, the switch between regimes is governed by \( y_{t-1} > s \), and \( l \) and \( s \) are estimated along with other parameters of the model. In such a case, under the null hypothesis of a single regime, the parameters \( l \) and \( s \) are not identified. Standard asymptotic distribution theory does not go through (Hansen, 1996). In our paper, since we do not estimate the parameters \( l \) and \( s \) but instead set \( l = 2 \) and \( s = 0 \) in accordance with the findings in previous studies, our tests do not suffer from this problem. Our test here is, therefore, carried out with the LR test statistic and critical values are drawn from the \( \chi^2 \) distribution with appropriate degrees of freedom.

The likelihood ratio (LR) test statistic, reported in Table 4, shows a p-value of 0.054 for the test. Thus, linearity is not rejected (barely) at the 5 percent significance level, but can be rejected at the 10 percent level.
Figure 3a plots the model-implied P/D ratios in the asset pricing model with SETAR(1,1) endowment. The P/D ratios are computed using the approximate solution method in Tauchen and Hussey (1991), after approximating the SETAR(1,1) process with discrete Markov chains using the techniques in Tauchen (1986). The mean P/D ratio is once again about 28.55, the value obtained under AR(1) endowment.

Figure 3b plots the P/D ratios under SETAR(1,1) endowment, along with their exact values obtained under AR(1) endowment. The figure shows that, while small differences are noticeable, the variation in the P/D ratios under SETAR(1,1) endowment is largely similar to that under AR(1) endowment. SETAR(1,1) minus exact AR(1) P/D ratios, plotted in Figure 3c, show a maximum difference of about 0.12. Compared to the mean P/D ratio of 28.55, this difference is quite small.

Figure 4 plots the same quantities as in Figure 3, except that approximate P/D ratios are used in the AR(1) case in place of their exact values. Comparing Figures 3b and 3c with Figures 4b and 4c, respectively, suggests largely similar behavior. It is worth recalling that the maximum approximation error in the implied P/D ratios under AR(1) endowment is 0.003. Given that the maximum discrepancy between the implied P/D ratios under SETAR(1,1) and AR(1) endowments is 0.12, this approximation error is unimportant in evaluating the differences in implications of non-linear and linear endowment processes.

6. Implications of Non-Linearities for Model-Implied Rates of Return

Given the model-implied P/D ratios, it is straightforward to compute the model-implied rate of return on risky assets with both linear and non-linear endowment. This
will not be derived here but the reader can refer, for instance, to Bidarkota and
McCulloch (2003). Similarly, the rate of return on risk free assets can also be easily
evaluated. However, while an exact closed-form solution can be easily derived for the
risk free rate with linear endowment, such a solution is not tractable with non-linear
endowment. The discretization of the state vector involved in the Tauchen-Hussey (1991)
numerical solution permits computation of the risk free returns with non-linear
endowment.

Table 5 presents the means and standard deviations of the model-implied rates of
return with both SETAR(1,1) and AR(1) endowments. The table reports statistics for the
implied risk free returns, equity returns, and equity premia. The model predicts a mean
risk free rate of 17.31 percent per annum with the SETAR(1,1) endowment and an equity
return of 17.36 percent per annum. Thus, the equity premium predicted by the model is
only 0.05 percent per annum. This low equity premium, as compared to the U.S. average
in excess of 7 percent per annum over the last 100 years, is the well-known equity

The model with a linear AR(1) endowment implies mean rates of return that are
within 0.01 percent of those from the model with the non-linear SETAR(1,1) endowment.
The approximation errors on the rates of return in the linear AR(1) case associated with
the numerical solution techniques of Tauchen (1986) and Tauchen and Hussey (1991) are
even smaller, as evident from the first two rows of Table 5. The biggest impact of the
approximate solution appears to be in the standard deviation of the equity premium. The
model with linear endowment implies a standard deviation of the equity premium of 3.07
percent per annum with the exact solution and only 1.76 with the numerical solution.
Overall, it appears that in the context of the consumption-based asset-pricing model of Lucas (1978), the effects of modeling non-linearities in the endowment process on the model-implied rates of return are miniscule. The mean implied rates of return in the model with linear and non-linear endowment are within 0.01 percent per annum of each other.

7. Conclusions

In this study we analyzed the economic importance of modeling non-linearities in the conditional mean dynamics of exogenous forcing processes on the moments of endogenous variables in the context of a macroeconomic model. We used the popular consumption-based asset pricing model of Lucas (1978) for our analysis. A linear AR(1) process and a non-linear SETAR(1,1) process were used alternatively to model the exogenous endowment sequence. Exact solution to the model with a linear endowment was used to solve for endogenous quantities of interest, including the model-implied price-dividend ratio, risk free and risky rates of return. A numerical solution due to Tauchen and Hussey (1991) was used to solve the model with a non-linear endowment.

Our analysis suggests that the overall impact of modeling non-linearities in the conditional mean dynamics of the exogenous endowment sequence on the implied rates of return in this model is miniscule. Differences in the mean rates of return implied by the model with linear and non-linear endowment processes are less than 0.01 percent per annum.

Our finding of weak effects of non-linearities is in part due to weak non-linearities in the conditional mean dynamics of GNP data, as noted in section 5. It is
conceivable that in other contexts where the underlying non-linearities are stronger, the overall impact of such non-linearities on the implications of models may be stronger.

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Figure 1. Plot of the Endowment Data

Fig.1 U.S. Real GNP growth rates
Figure 2. Asset Pricing Solution under Linear AR(1) Endowment

Exact and Approximate Solutions
Figure 3. Asset Pricing Solution under Non-Linear SETAR(1,1) Endowment

Comparison with Exact Solution under Linear AR(1) Endowment
Figure 4. Asset Pricing Solution under Non-Linear SETAR(1,1) Endowment

Comparison with Approximate Solution under Linear AR(1) Endowment
Table 1: Assessing the Accuracy of Markov Chain Approximations

\[ y_t = \mu + \rho(y_{t-1} - \mu) + \eta_t, \quad \eta_t \sim \text{iid } N(0, \sigma^2_{\eta}). \quad (11) \]

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<th>(\mu)</th>
<th>(\rho)</th>
<th>(\sigma^2_{\eta})</th>
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<td>2.5</td>
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Notes to Table 1:
1. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.
Table 2: Assessing the Accuracy of Markov Chain Approximations

In regime 1, \( y_t = \mu_1 + \rho_1 (y_{t-1} - \mu_1) + \eta_{1t}, \) \( \eta_{1t} \sim \text{iid } N(0, \sigma^2_{\eta_1}) \) (15.1)

In regime 2, \( y_t = \mu_2 + \rho_2 (y_{t-1} - \mu_2) + \eta_{2t}, \) \( \eta_{2t} \sim \text{iid } N(0, \sigma^2_{\eta_2}) \) (15.2)

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Notes to Table 2:
1. When \( y_{t-2} > 0, \) we get regime 1 and when \( y_{t-2} \leq 0, \) we get regime 2.
2. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.
Table 3: Maximum Likelihood Model Estimates for Linear AR(1) Process

\[ y_t = \mu + \rho (y_{t-1} - \mu) + \eta_t, \quad \eta_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\eta}). \]  

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<td>(0.0011)</td>
<td>(0.0005)</td>
<td>(0.0005)</td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table 3:
1. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.
2. log L is the maximized log-likelihood value.
Table 4: Maximum Likelihood Model Estimates for Non-Linear SETAR(1,1) Process

In regime 1,  
\[ y_t = \mu_1 + \rho_1 (y_{t-1} - \mu_1) + \eta_{1t}, \quad \eta_{1t} \sim \text{iid } N(0, \sigma^2_{\eta_1}) \]  
(15.1)

In regime 2,  
\[ y_t = \mu_2 + \rho_2 (y_{t-1} - \mu_2) + \eta_{2t}, \quad \eta_{2t} \sim \text{iid } N(0, \sigma^2_{\eta_2}) \]  
(15.2)

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>( \rho_1 )</th>
<th>( \sigma^2_{\eta_1} )</th>
<th>( \mu_2 )</th>
<th>( \rho_2 )</th>
<th>( \sigma^2_{\eta_2} )</th>
<th>( \log L )</th>
<th>( 2\Delta \log L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0080</td>
<td>0.3843</td>
<td>0.0093</td>
<td>0.0072</td>
<td>0.2742</td>
<td>0.0129</td>
<td>631.275</td>
<td>7.636</td>
</tr>
<tr>
<td>(0.0012)</td>
<td>(0.0730)</td>
<td>(0.0005)</td>
<td>(0.0031)</td>
<td>(0.1731)</td>
<td>(0.0015)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table 4:

1. When \( y_{t-2} > 0 \), we get regime 1 and when \( y_{t-2} \leq 0 \), we get regime 2.
2. Numbers in parentheses for the parameter estimates are the Hessian-based standard errors.
3. \( \log L \) is the maximized log-likelihood value.
4. \( 2\Delta \log L \) is the likelihood ratio test statistic for the null hypothesis of a single regime. In this case, \( \mu_1 = \mu_2 \), \( \rho_1 = \rho_2 \), and \( \sigma^2_{\eta_1} = \sigma^2_{\eta_2} \). The number in parentheses gives the \( \chi^2_3 \) p-value.
### Table 5: Model-Implied Rates of Return

<table>
<thead>
<tr>
<th></th>
<th>Risk free returns</th>
<th>Equity returns</th>
<th>Equity premium</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AR(1) Exact</strong></td>
<td>17.3197</td>
<td>17.3629</td>
<td>0.0432</td>
</tr>
<tr>
<td></td>
<td>2.4057</td>
<td>3.9001</td>
<td>3.0703</td>
</tr>
<tr>
<td><strong>AR(1) Approximate</strong></td>
<td>17.3140</td>
<td>17.3630</td>
<td>0.0491</td>
</tr>
<tr>
<td></td>
<td>2.3949</td>
<td>3.9018</td>
<td>1.7567</td>
</tr>
<tr>
<td><strong>SETAR(1,1) Approximate</strong></td>
<td>17.3089</td>
<td>17.3592</td>
<td>0.0503</td>
</tr>
<tr>
<td></td>
<td>2.4225</td>
<td>3.8852</td>
<td>1.7513</td>
</tr>
</tbody>
</table>

**Notes to Table 5:**

1. For each of the three asset pricing solutions in the first column, numbers in the first row of the remaining columns give the mean rates of return and the numbers in the second row of the remaining columns give the standard deviations of the rates of return.
2. All rates of return are expressed in percent per annum.